# recurrence limits

1. Two sequences are generated by the recurrence relations $u_{n+1} = au_n + 10$ and $v_{n+1} = a^2 v_n + 16$.

The two sequences approach the same limit as $n \to \infty$.

Determine the value of $a$ and evaluate the limit. **5**

2. A sequence is defined by the recurrence relation $u_{n+1} = 0.3u_n + 5$ with first term $u_1$.

(a) Explain why this sequence has a limit as $n$ tends to infinity. **1**

(b) Find the **exact** value of this limit. **2**

3. Two sequences are defined by these recurrence relations:

$$u_{n+1} = 3u_n - 0.4 \text{ with } u_0 = 1, \qquad v_{n+1} = 0.3v_n + 4 \text{ with } v_0 = 1.$$

(a) Explain why only one of these sequences approaches a limit as $n \to \infty$. **1**

(b) Find algebraically the exact value of the limit. **2**

(c) For the other sequence, find

(i) the smallest value of $n$ for which the $n^{th}$ term exceeds 1000, and

(ii) the value of that term. **2**

4. A man decides to plant a number of fast-growing trees as a boundary between his property and the property of his next door neighbour. He has been warned, however, by the local garden centre that, during any year, the trees are expected to increase in height by 0·5 metres. In response to this warning he decides to trim 20% off the height of the trees at the start of any year.

(a) If he adopts the "20% pruning policy", to what height will he expect the trees to grow in the long run? **3**

(b) His neighbour is concerned that the trees are growing at an alarming rate and wants assurances that the trees will grow no taller than 2 metres. What is the minimum percentage that the trees will need to be trimmed each year so as to meet this condition. **3**

5. (a) A sequence is defined by $u_{n+1} = -\frac{1}{2}u_n$ with $u_0 = -16$.

   Write down the values of $u_1$ and $u_2$.     **1**

   (b) A second sequence is given by $4, 5, 7, 11, \ldots$.

   It is generated by the recurrence relation $v_{n+1} = pv_n + q$ with $v_1 = 4$.

   Find the values of $p$ and $q$.     **3**

   (c) Either the sequence in (a) or the sequence in (b) has a limit.
       (i) Calculate this limit.
       (ii) Why does the other sequence not have a limit?     **3**

6. Secret Agent 004 has been captured and his captors are giving him a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour.

   (a) Calculate how many milligrams of serum remain in his body after 4 hours (that is immediately **before** the second dose is given).     **(3)**

   (b) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Find how many doses are needed before the captors should begin their interrogation.     **(3)**

   (c) Let $u_n$ be the amount of serum (in milligrams) in his body just **after** his $n^{\text{th}}$ dose. Show that $u_{n+1} = 0 \cdot 522 u_n + 25$.     **(1)**

   (d) It is also known that 55 milligrams of this serum in the body will prove fatal, and the captors wish to keep Agent 004 alive. Is there any maximum length of time for which they can continue to administer this serum and still keep him alive?     **(4)**

7. A gardener feeds her trees weekly with "Bioforce, the wonder plant food".
It is known that in a week the amount of plant food in the tree falls by
about 25%.

    (*a*)   The trees contain no Bioforce initially and the gardener applies 1g of
           Bioforce to each tree every Saturday. Bioforce is only effective when there
           is continuously more than 2g of it in the tree. Calculate how many weekly
           feeds will be necessary before the Bioforce becomes effective. **(3)**

    (*b*)   (i)   Write down a recurrence relation for the amount of plant food in the
                tree immediately after feeding. **(1)**
           (ii)  If the level of Bioforce in the tree exceeds 5g, it will cause leaf burn.
                Is it safe to continue feeding the trees at this rate indefinitely? **(4)**

8. Trees are sprayed weekly with the pesticide, KILLPEST, whose manufacturers
claim it will destroy 65% of all pests. Between the weekly sprayings it is estimated
that 500 new pests invade the trees.

A new pesticide, PESTKILL, comes onto the market. The manufacturers claim that
it will destroy 85% of existing pests but it is estimated that 650 new pests per week
will invade the trees.

Which pesticide will be more effective in the long term? **7**

9. Some environmentalists are concerned that the presence of chemical nitrates in drinking water presents a threat to health.

The World Health Organisation recommends an upper limit of 50 milligrams per litre (mg/l) for nitrates in drinking water, although it regards levels up to 100 mg/l as safe.

A sub-committee of a Local Water Authority is considering a proposal affecting a small loch which supplies a nearby town with drinking water. The proposal is that a local factory be permitted to make a once-a-week discharge of effluent into the loch, provided that a cleaning treatment of the loch is carried out before each discharge of effluent.

The Water Engineer has presented the following data:

> 1. The present nitrate level in the loch is 20 mg/l.
> 2. The cleaning treatment removes 55% of the nitrates from the loch.
> 3. Each discharge of effluent will result in an addition of 26 mg/l to the nitrate presence in the loch.

and advises the sub-committee that the proposal presents no long-term danger from nitrates to the drinking water supply.

(a) Show the calculations you would use to check the engineer's advice. **(5)**

(b) Is the engineer's advice acceptable? **(1)**

10. Two sequences are defined by the recurrence relations

$$u_{n+1} = 0.2u_n + p, \quad u_0 = 1 \quad \text{and}$$
$$v_{n+1} = 0.6v_n + q, \quad v_0 = 1.$$

If both sequences have the same limit, express $p$ in terms of $q$. **3**

11. A sequence is defined by the recurrence relation $u_n = 0.9u_{n-1} + 2$, $u_1 = 3$.

(a) Calculate the value of $u_2$. **1**

(b) What is the smallest value of $n$ for which $u_n > 10$? **1**

(c) Find the limit of this sequence as $n \to \infty$. **2**

12. (a) At 12 noon a hospital patient is given a pill containing 50 units of antibiotic.
By 1 pm the number of units in the patient's body has dropped by 12%.
By 2 pm a further 12% of the units remaining in the body at 1 pm is lost.
If this fall-off rate is maintained, find the number of units of antibiotic remaining at 6 pm. **(4)**

(b) A doctor considers prescribing a course of treatment which involves a patient taking one of these pills every 6 hours over a long period of time.
The doctor knows that mopre than 100 units of this antibiotic in the body is regarded as too dangerous.
Should the doctor prescribe this course of treatment?
Give reasons for your answer. **(6)**

13. Biologists calculate that when the concentration of a particular chemical in a sea loch reaches 5 milligrams per litre (mg/l) the level of pollution endangers the life of the fish.
A factory wishes to release waste containing this chemical into the loch. It is claimed that the discharge will not endanger the fish.
The Local Authority is supplied with the following information:

1. The loch contains none of this chemical at present.
2. The factory manager has applied to discharge effluent once per week which will result in an increase in concentration of 2.5 mg/l of the chemical in the loch.
3. The natural tidal action will remove 40% of the chemical from the loch every week.

(a) Show that this level of discharge would result in fish being endangered. **(3)**

When this result is announced, the company agrees to install a cleaning process that reduces the concentration of chemical released into the loch by 30%.

(b) Show the calculations you would use to check this revised application.
Should the Local Authority grant permission ? **(5)**

*[END OF QUESTIONS]*